

A generalization of Istrăţescu's fixed point theorem for convex contractions

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Abstract. In this paper we prove a generalization of Istrăţescu's theorem for convex contractions. More precisely, we introduce the concept of iterated function system consisting of convex contractions and prove the existence and uniqueness of the attractor of such a system. In addition we study the properties of the canonical projection from the code space into the attractor of an iterated function system consisting of convex contractions.

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1. Introduction

Banach-Caccioppoli-Picard contraction principle, which is an extremely useful tool in nonlinear analysis, says that any contraction $f : (X, d) \rightarrow (X, d)$, where (X, d) is a complete metric space, has a unique fixed point x^* and $\lim_{n \rightarrow \infty} f^{[n]}(x) = x^*$ for every $x \in X$. Besides its great features (the uniqueness of the fixed point and the possibility to approximate it by the means of Picard iteration) there exists a drawback of this result, namely that the contraction condition is too strong.

The natural question if there exist contraction-type conditions that do not imply the contraction condition and for which the existence and uniqueness of the fixed point are assured was answered, among others, by V. Istrăţescu who introduced and studied the convex contraction condition (see [5], [6] and [7]). More precisely, a continuous function $f : (X, d) \rightarrow (X, d)$, where (X, d) is a complete metric space, is called convex contraction if there exist $a, b \in (0, 1)$ such that $a + b < 1$ and $d(f^{[2]}(x), f^{[2]}(y)) \leq ad(f(x), f(y)) + bd(x, y)$ for every $x, y \in X$. Istrăţescu proved that any convex contraction has a unique fixed point $x^* \in X$ (and $\lim_{n \rightarrow \infty} f^{[n]}(x) = x^*$ for every $x \in X$) and provided an example of convex contraction which is not contraction. V. Ghorbanian, S. Rezapour and N. Shahzad [8] generalized Istrăţescu's results to complete ordered metric spaces. M. A. Miandaragh, M. Postolache and S. Rezapour [16] introduced the concept of generalized convex contraction and proved some

theorems about approximate fixed points of these contractions. Extending these results, A. Latif, W. Sintunavarat and A. Ninsri [12] introduced a new concept called partial generalized convex contraction and established some approximate fixed point results for such mappings in α -complete metric spaces. For more results along these lines of generalization one can also see [10].

Let us recall that an iterated function system on a complete metric space (X, d) , denoted by $\mathcal{S} = (X, (f_k)_{k \in \{1, 2, \dots, n\}})$, consists of a finite family of contractions $(f_k)_{k \in \{1, 2, \dots, n\}}$, where $f_k : X \rightarrow X$. The function $F_{\mathcal{S}} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ defined by $F_{\mathcal{S}}(C) = \bigcup_{k=1}^n f_k(C)$, for all $C \in \mathcal{K}(X)$ -the set of non-empty compact subsets of X -, which is called the set function associated to \mathcal{S} , turns out to be a contraction (with respect to the Hausdorff-Pompeiu distance) and its unique fixed point, denoted by $A_{\mathcal{S}}$, is called the attractor of the system \mathcal{S} . As iterated function systems represent one of the main tools to generate fractals, the extending problem of the notion of iterated function system was treated by several authors. Let us mention some contributions along these lines of research. Given a complete metric space (X, d) and a finite family of functions $f_1, f_2, \dots, f_n : X \rightarrow X$, L. Máté [15] proved the existence of a unique $A \in \mathcal{K}(X)$ such that $A = \bigcup_{i=1}^n f_i(A)$ under weaker contractivity conditions (for example $d(f_i(x), f_i(y)) \leq \varphi(d(x, y))$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an upper continuous non-decreasing function with the property that $\varphi(t) < t$ for each $t > 0$). K. Leśniak [13] presented a multivalued approach of infinite iterated function systems. A. Petruşel [21] proved that each finite family of single-valued and multi-valued operators satisfying some Meir-Keeler type conditions has a self-similar set (see also [4]). Let (X, d) be a metric space and $f_1, f_2, \dots, f_n : X \rightarrow P_{cl}(X)$ be set-valued mappings on X , where $P_{cl}(X)$ designates the family of all nonempty closed subsets of X . The system $F = (f_1, f_2, \dots, f_n)$ is called an iterated multifunction system and the operator $\hat{F} : P_{cl}(X) \rightarrow P_{cl}(X)$ given by $\hat{F}(Y) = \overline{\bigcup_{i=1}^n f_i(Y)}$, where $f_i(Y) = \bigcup_{x \in Y} f_i(x)$ for each $i \in \{1, 2, \dots, n\}$, is called the Barnsley-Hutchinson operator generated by F . A fixed point of this operator is called a multivalued large fractal. C. Chifu and A. Petruşel [3] obtained existence and uniqueness results for multivalued large fractals (see also [20]). G. Gwóźdź-Lukowska and J. Jachymski [9] developed the Hutchinson-Barnsley theory for finite families of mappings on a metric space endowed with a directed graph. E. Llorens-Fuster, A. Petruşel and J.-C. Yao [14] gave existence and uniqueness results for self-

similar sets of a mixed iterated function system. M. Boriceanu, M. Bota and A. Petruşel [2] extended the Hutchinson-Barnsley theory to the case of set-valued mappings on a b -metric space. For other related results see [1], [11], [17], [19], [22], [24], [25] and [26].

In this paper we introduce the concept of iterated function system consisting of convex contractions and prove the existence and uniqueness of the attractor of such a system obtaining in this way a generalization of Istrăţescu's convex contractions fixed point theorem (see Theorem 3.2). Moreover we study the properties of the canonical projection from the code space into the attractor of an iterated function system consisting of convex contractions (see Theorem 3.6).

2. Preliminaries

Given a function $f : X \rightarrow X$, by $f^{[n]}$ we mean the composition of f by itself n times.

Given a set X and a family of functions $(f_i)_{i \in I}$, where $f_i : X \rightarrow X$, by $f_{\alpha_1 \alpha_2 \dots \alpha_n}$ we mean $f_{\alpha_1} \circ f_{\alpha_2} \circ \dots \circ f_{\alpha_n}$ and by $Y_{\alpha_1 \alpha_2 \dots \alpha_n}$ we understand $f_{\alpha_1 \alpha_2 \dots \alpha_n}(Y)$, where $Y \subseteq X$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in I$.

Given a set X , by $\mathcal{P}^*(X)$ we denote the family of all nonempty subsets of X . For a metric space (X, d) , by $\mathcal{K}(X)$ we denote the set of non-empty compact subsets of X .

Given two sets A and B , by B^A we mean the set of functions from A to B .

Given a set I , $\Lambda(I)$ denotes $I^{\mathbb{N}^*}$ and $\Lambda_n(I)$ denotes $I^{\{1, 2, \dots, n\}}$. Hence the elements of $\Lambda(I)$ can be written as infinite words $\alpha = \alpha_1 \alpha_2 \alpha_3 \dots$ and the elements of $\Lambda_n(I)$ as finite words $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$. By $\Lambda^*(I)$ we denote the set of all finite words, namely $\Lambda^*(I) = \bigcup_{n \in \mathbb{N}^*} \Lambda_n(I) \cup \{\lambda\}$, where λ is the empty word. $\Lambda(I)$ can be seen as a metric space with the distance d_Λ defined by $d_\Lambda(\alpha, \beta) = \frac{1}{2^n}$ where n is the natural number having the property that $\alpha_k = \beta_k$ for $k < n$ and $\alpha_n \neq \beta_n$ if $\alpha = \alpha_1 \alpha_2 \alpha_3 \dots \alpha_n \alpha_{n+1} \dots \neq \beta = \beta_1 \beta_2 \beta_3 \dots \beta_n \beta_{n+1} \dots$ and $d_\Lambda(\alpha, \alpha) = 0$. By $\alpha\beta$ we understand the concatenation of the words $\alpha \in \Lambda^*$ and $\beta \in \Lambda \cup \Lambda^*$. For $\alpha \in \Lambda \cup \Lambda_n$ and $m \leq n$, $[\alpha]_m \stackrel{def}{=} \alpha_1 \alpha_2 \dots \alpha_m$. For $i \in I$, let us consider the function $F_i : \Lambda(I) \rightarrow \Lambda(I)$ given by $F_i(\alpha) = i\alpha$ for all $\alpha \in \Lambda(I)$.

Definition 2.1. For a metric space (X, d) , we consider on $\mathcal{P}^*(X)$ the generalized Hausdorff-Pompeiu pseudometric $h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \rightarrow [0, +\infty]$ defined by

$$h(A, B) = \max(d(A, B), d(B, A)) = \inf\{\eta \in [0, \infty] \mid A \subseteq N_\eta(B) \text{ and } B \subseteq N_\eta(A)\}$$

where

$$d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} (\inf_{y \in B} d(x, y))$$

and

$$N_\eta(A) = \{x \in X \mid \text{there exists } y \in A \text{ such that } d(x, y) < \eta\},$$

for every $A, B \in \mathcal{P}^*(X)$.

Proposition 2.2 (see [23]). If H and K are two nonempty subsets of the metric space (X, d) , then

$$h(H, K) = h(\overline{H}, \overline{K}).$$

Proposition 2.3 (see [23]). If $(H_i)_{i \in I}$ and $(K_i)_{i \in I}$ are two families of nonempty subsets of the metric space (X, d) , then

$$h(\bigcup_{i \in I} H_i, \bigcup_{i \in I} K_i) = h(\overline{\bigcup_{i \in I} H_i}, \overline{\bigcup_{i \in I} K_i}) \leq \sup_{i \in I} h(H_i, K_i).$$

Theorem 2.4 (see [23]). If the metric space (X, d) is complete, then $(\mathcal{K}(X), h)$ is a complete metric space.

Definition 2.5. For a metric space (X, d) , we consider on $\mathcal{P}^*(X)$ the function $\delta : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \rightarrow [0, +\infty]$ defined by

$$\delta(A, B) = \sup_{x \in A, y \in B} d(x, y),$$

for all $A, B \in \mathcal{P}^*(X)$.

Remark 2.6. For every $A, B \in \mathcal{P}^*(X)$ we have

$$h(A, B) \leq \delta(A, B).$$

Proposition 2.7. *Let (X, d) be a complete metric space, $(Y_n)_{n \in \mathbb{N}} \subseteq \mathcal{K}(X)$ and Y a closed subset of X such that $\lim_{n \rightarrow \infty} h(Y_n, Y) = 0$. Then $Y \in \mathcal{K}(X)$.*

Proof. It is enough to prove that Y is precompact. To this aim, let us note that for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $h(Y_{n_\varepsilon}, Y) < \frac{\varepsilon}{2}$, so $Y \subseteq N_{\frac{\varepsilon}{2}}(Y_{n_\varepsilon})$. Since $Y_{n_\varepsilon} \in \mathcal{K}(X)$ there exist $x_1, \dots, x_m \in X$ such that $Y_{n_\varepsilon} \subseteq \bigcup_{i=1}^m B(x_i, \frac{\varepsilon}{2})$ and therefore $Y \subseteq \bigcup_{i=1}^m B(x_i, \varepsilon)$. \square

Proposition 2.8. *Let (X, d) be a complete metric space, $(Y_n)_{n \in \mathbb{N}} \subseteq \mathcal{K}(X)$ and $Y \in \mathcal{K}(X)$ such that $\lim_{n \rightarrow \infty} h(Y_n, Y) = 0$. Then $H \stackrel{\text{def}}{=} Y \cup \left(\bigcup_{n=0}^{\infty} Y_n \right) \in \mathcal{K}(X)$.*

Proof. First of all we prove that H is a closed subset of X .

Indeed, if $x \in \overline{H}$, then there exists a sequence $(x_k)_{k \in \mathbb{N}} \subseteq H$ such that $\lim_{k \rightarrow \infty} x_k = x$.

If $\{k \in \mathbb{N} \mid x_k \in Y\}$ is infinite, then there exists a subsequence $(x_{k_p})_{p \in \mathbb{N}}$ of $(x_k)_{k \in \mathbb{N}}$ such that $x_{k_p} \in Y$ for every $p \in \mathbb{N}$. Since $Y \in \mathcal{K}(X)$ there exists a subsequence $(x_{k_{pq}})_{q \in \mathbb{N}}$ of $(x_{k_p})_{p \in \mathbb{N}}$ and $y \in Y$ such that $\lim_{q \rightarrow \infty} x_{k_{pq}} = y$. Consequently, as $\lim_{q \rightarrow \infty} x_{k_{pq}} = y$, we conclude that $x = y \in Y \subseteq H$.

If there exists $n_0 \in \mathbb{N}$ such that $\{k \in \mathbb{N} \mid x_k \in Y_{n_0}\}$ is infinite, a similar argument shows that $x \in H$.

If none of the above described two cases is valid, then there exist an increasing sequence $(k_p)_{p \in \mathbb{N}} \subseteq \mathbb{N}$, $x_{k_p} \in Y_{k_p}$ and $y_{k_p} \in Y$ such that

$$d(x_{k_p}, y_{k_p}) < h(Y_{k_p}, Y) + \frac{1}{p}.$$

Since $Y \in \mathcal{K}(X)$ there exists $(y_{k_{pq}})_{q \in \mathbb{N}}$ a subsequence of $(y_{k_p})_{p \in \mathbb{N}}$ and $y \in Y$ such that $\lim_{q \rightarrow \infty} y_{k_{pq}} = y$. As

$$d(x_{k_{pq}}, y) < d(x_{k_{pq}}, y_{k_{pq}}) + d(y_{k_{pq}}, y) \leq h(Y_{k_{pq}}, Y) + \frac{1}{p_q} + d(y_{k_{pq}}, y)$$

and

$$\lim_{q \rightarrow \infty} h(Y_{k_{pq}}, Y) = \lim_{q \rightarrow \infty} \frac{1}{p_q} = \lim_{q \rightarrow \infty} d(y_{k_{pq}}, y) = 0,$$

we infer that $\lim_{q \rightarrow \infty} x_{k_{pq}} = y$. Consequently, as $\lim_{q \rightarrow \infty} x_{k_{pq}} = x$, we get $x = y \in Y \subseteq H$.

Now we prove that

$$\lim_{m \rightarrow \infty} h(\bigcup_{i=0}^m Y_i, H) = 0.$$

Indeed

$$\begin{aligned} h(\bigcup_{i=0}^m Y_i, H) &= h((\bigcup_{i=0}^m Y_i) \cup (\bigcup_{i=m+1}^{\infty} Y_m) \cup Y_m, (\bigcup_{i=0}^m Y_i) \cup (\bigcup_{i=m+1}^{\infty} Y_i) \cup Y) \stackrel{\text{Proposition 2.3}}{\leq} \\ &\leq \sup\{h(Y_m, Y), h(Y_m, Y_{m+1}), h(Y_m, Y_{m+2}), \dots\} \end{aligned}$$

for every $m \in \mathbb{N}$. As $\lim_{m \rightarrow \infty} h(Y_m, Y) = 0$, we conclude that $\lim_{m \rightarrow \infty} h(\bigcup_{i=0}^m Y_i, H) = 0$.

Because $\bigcup_{i=0}^m Y_i \in \mathcal{K}(X)$ for every $m \in \mathbb{N}$ and H is closed, using Proposition 2.7, we obtain that H is compact. \square

3. The main results

Definition 3.1. *An iterated function system consisting of convex contractions on a complete metric space (X, d) is given by a finite family of continuous functions $(f_i)_{i \in I}$, $f_i : X \rightarrow X$, such that for every $i, j \in I$ there exist $a_{ij}, b_{ij}, c_{ij} \in [0, \infty)$ satisfying the following two properties:*

- $\alpha)$ $a_{ij} + b_{ij} + c_{ij} \stackrel{\text{def}}{=} d_{ij}$ and $\max_{i, j \in I} d_{ij} \stackrel{\text{def}}{=} d < 1$;
- $\beta)$

$$d((f_i \circ f_j)(x), (f_i \circ f_j)(y)) \leq a_{ij}d(x, y) + b_{ij}d(f_i(x), f_i(y)) + c_{ij}d(f_j(x), f_j(y))$$

for every $i, j \in I$ and every $x, y \in X$.

We denote such a system by

$$\mathcal{S} = ((X, d), (f_i)_{i \in I}).$$

One can associate to the system \mathcal{S} the function $F_{\mathcal{S}} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ given by

$$F_{\mathcal{S}}(B) = \bigcup_{i \in I} f_i(B)$$

for all $B \in \mathcal{K}(X)$.

Theorem 3.2. *Let $\mathcal{S} = ((X, d), (f_i)_{i \in I})$ be an iterated function system consisting of convex contractions. Then:*

i) *There exists a unique $A \in \mathcal{K}(X)$ such that*

$$\lim_{n \rightarrow \infty} h(F_{\mathcal{S}}^{[n]}(B), A) = 0,$$

for every $B \in \mathcal{K}(X)$.

ii) *For each $\omega \in \Lambda(I)$ there exists $a_{\omega} \in X$ such that*

$$\lim_{n \rightarrow \infty} h(f_{[\omega]_n}(B), \{a_{\omega}\}) = 0,$$

for every $B \in \mathcal{K}(X)$.

Moreover

$$\lim_{n \rightarrow \infty} \sup_{\omega \in \Lambda(I)} h(f_{[\omega]_n}(B), \{a_{\omega}\}) = 0$$

for every $B \in \mathcal{K}(X)$.

iii)

$$A = \overline{\{a_{\omega} \mid \omega \in \Lambda(I)\}}.$$

iv) *For every $(Y_n)_{n \in \mathbb{N}} \subseteq \mathcal{K}(X)$ and $Y \in \mathcal{K}(X)$, the following implication is valid:*

$$\lim_{n \rightarrow \infty} h(Y_n, Y) = 0 \Rightarrow \lim_{n \rightarrow \infty} h(F_{\mathcal{S}}(Y_n), F_{\mathcal{S}}(Y)) = 0.$$

v) *A is the unique fixed point of $F_{\mathcal{S}}$.*

Proof.

i) For fixed $Y, Z \in \mathcal{K}(X)$ we define

$$x_n(Y, Z) = \sup_{\omega \in \Lambda_n(I)} \delta(f_{\omega}(Y), f_{\omega}(Z))$$

and

$$y_n(Y, Z) = \max\{x_{n-1}(Y, Z), x_n(Y, Z)\}$$

for every $n \in \mathbb{N}^*$. For the sake of simplicity we will denote $x_n(Y, Z)$ by x_n and $y_n(Y, Z)$ by y_n .

We claim that the sequence $(y_n)_{n \in \mathbb{N}^*}$ is decreasing.

Indeed, for $n \in \mathbb{N}^*$ and $\omega \in \Lambda_{n+1}(I)$ there exist $i, j \in I$ and $\omega_0 \in \Lambda_{n-1}(I)$ such that $\omega = ij\omega_0$. Then, for $y \in Y$ and $z \in Z$, we have

$$\begin{aligned} d(f_{\omega}(y), f_{\omega}(z)) &= d(f_{ij\omega_0}(y), f_{ij\omega_0}(z)) \leq \\ &\leq a_{ij}d(f_{\omega_0}(y), f_{\omega_0}(z)) + b_{ij}d(f_{i\omega_0}(y), f_{i\omega_0}(z)) + c_{ij}d(f_{j\omega_0}(y), f_{j\omega_0}(z)) \leq \end{aligned}$$

$$\begin{aligned}
&\leq a_{ij}x_{n-1} + b_{ij}x_n + c_{ij}x_n \leq a_{ij}x_{n-1} + (b_{ij} + c_{ij})x_n \leq \\
&\leq d_{ij} \max\{x_{n-1}, x_n\} = d_{ij}y_n \leq dy_n < y_n,
\end{aligned}$$

so

$$x_{n+1} = \sup_{\omega \in \Lambda_{n+1}(I)} \delta(f_\omega(Y), f_\omega(Z)) \leq dy_n < y_n. \quad (1)$$

As

$$x_n \leq \max\{x_{n-1}, x_n\} = y_n, \quad (2)$$

we get

$$y_{n+1} = \max\{x_n, x_{n+1}\} \stackrel{(1) \text{ and } (2)}{\leq} y_n.$$

Therefore we have

$$y_{n+2} = \max\{x_{n+1}, x_{n+2}\} \stackrel{(1)}{\leq} \max\{dy_n, dy_{n+1}\} = dy_n$$

and consequently

$$y_{2n-1} \leq d^{n-1}y_1$$

and

$$y_{2n} \leq d^{n-1}y_1$$

for every $n \in \mathbb{N}^*$.

Thus the series $\sum_{n \in \mathbb{N}^*} y_n$ is convergent, so the series $\sum_{n \in \mathbb{N}^*} x_n$ is convergent (see (2) and use the comparison test) and consequently $\lim_{n \rightarrow \infty} x_n = 0$. Hence, as

$$\begin{aligned}
h(F_S^{[n]}(Y), F_S^{[n]}(Z)) &= h\left(\bigcup_{\omega \in \Lambda_n(I)} f_\omega(Y), \bigcup_{\omega \in \Lambda_n(I)} f_\omega(Z)\right) \stackrel{\text{Proposition 2.3}}{\leq} \\
&\leq \sup_{\omega \in \Lambda_n(I)} h(f_\omega(Y), f_\omega(Z)) \stackrel{\text{Remark 2.6}}{\leq} \sup_{\omega \in \Lambda_n(I)} \delta(f_\omega(Y), f_\omega(Z)) = x_n \quad (3)
\end{aligned}$$

for every $n \in \mathbb{N}^*$, we get that

$$\lim_{n \rightarrow \infty} h(F_S^{[n]}(Y), F_S^{[n]}(Z)) = 0. \quad (4)$$

In particular, for each $Y \in \mathcal{K}(X)$, considering $Z = F_S(Y) \in \mathcal{K}(X)$ and taking into account the comparison test and (3), we infer that the series $\sum_{n \in \mathbb{N}^*} h(F_S^{[n+1]}(Y), F_S^{[n]}(Y))$ is convergent. Thus we conclude that the sequence

$(F_S^{[n+1]}(Y))_{n \in \mathbb{N}^*}$ is Cauchy and, as $(\mathcal{K}(X), h)$ is complete (see Theorem 2.4), there exists $A_Y \in \mathcal{K}(X)$ such that

$$\lim_{n \rightarrow \infty} h(F_S^{[n]}(Y), A_Y) = 0. \quad (5)$$

In the same manner we can prove that if $Z \in \mathcal{K}(X)$, then

$$\lim_{n \rightarrow \infty} h(F_S^{[n]}(Z), A_Z) = 0. \quad (6)$$

From (4), (5) and (6) we obtain that $A_Y = A_Z \stackrel{def}{=} A$ for every $Y, Z \in \mathcal{K}(X)$. Thus

$$\lim_{n \rightarrow \infty} h(F_S^{[n]}(B), A) = 0,$$

for every $B \in \mathcal{K}(X)$.

ii) For $\omega \in \Lambda(I)$ and $Y, Z \in \mathcal{K}(X)$ we have

$$h(f_{[\omega]_n}(Y), f_{[\omega]_n}(Z)) \stackrel{\text{Remark 2.6}}{\leq} \delta(f_{[\omega]_n}(Y), f_{[\omega]_n}(Z)) \leq \sup_{\omega \in \Lambda_n(I)} \delta(f_\omega(Y), f_\omega(Z)) = x_n$$

for every $n \in \mathbb{N}^*$, so, as $\lim_{n \rightarrow \infty} x_n = 0$, we deduce that

$$\lim_{n \rightarrow \infty} \delta(f_{[\omega]_n}(Y), f_{[\omega]_n}(Z)) = \lim_{n \rightarrow \infty} h(f_{[\omega]_n}(Y), f_{[\omega]_n}(Z)) = 0. \quad (7)$$

For $Y \in \mathcal{K}(X)$ we have

$$\begin{aligned} h(f_{[\omega]_n}(Y), f_{[\omega]_{n+1}}(Y)) &\stackrel{\text{Remark 2.6}}{\leq} \delta(f_{[\omega]_n}(Y), f_{[\omega]_{n+1}}(Y)) \leq \\ &\leq \delta(f_{[\omega]_n}(Y), f_{[\omega]_n}(F_S(Y))) \leq x_n(Y, F_S(Y)) \end{aligned}$$

for each $n \in \mathbb{N}^*$, hence, since -as we have seen in the proof of 1)- the series $\sum_{n \in \mathbb{N}^*} x_n(Y, F_S(Y))$ is convergent, using the comparison criterion, we infer that the series $\sum_{n \in \mathbb{N}^*} h(f_{[\omega]_n}(Y), f_{[\omega]_{n+1}}(Y))$ is convergent. Thus we conclude that the sequence $(f_{[\omega]_n}(Y))_{n \in \mathbb{N}^*}$ is Cauchy and as, $(\mathcal{K}(X), h)$ is complete (see Theorem 2.4), there exists $A_\omega(Y) \in \mathcal{K}(X)$ such that

$$\lim_{n \rightarrow \infty} h(f_{[\omega]_n}(Y), A_\omega(Y)) = 0. \quad (8)$$

In the same manner we can prove that if $Z \in \mathcal{K}(X)$, then there exists $A_\omega(Z) \in \mathcal{K}(X)$ such that

$$\lim_{n \rightarrow \infty} h(f_{[\omega]_n}(Z), A_\omega(Z)) = 0. \quad (9)$$

From (7), (8) and (9) we obtain that $A_\omega(Y) = A_\omega(Z) \stackrel{def}{=} A_\omega$ for each $Y, Z \in \mathcal{K}(X)$. Thus

$$\lim_{n \rightarrow \infty} h(f_{[\omega]_n}(B), A_\omega) = 0, \quad (10)$$

for each $B \in \mathcal{K}(X)$.

Since

$$\lim_{n \rightarrow \infty} \text{diam}(f_{[\omega]_n}(B)) = 0 \quad (11)$$

for each $B \in \mathcal{K}(X)$ (see (7) for $Y = Z = B$), we get that

$$\text{diam}(A_\omega) = 0. \quad (12)$$

Indeed, using (10) and (11), we infer that for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^*$ such that

$$\text{diam}(f_{[\omega]_{n_\varepsilon}}(B)) < \varepsilon \text{ and } h(f_{[\omega]_{n_\varepsilon}}(B), A_\omega) < \varepsilon.$$

Therefore there exists $\eta_0 \in (0, \varepsilon)$ such that

$$A_\omega \subseteq N_{\eta_0}(f_{[\omega]_{n_\varepsilon}}(B)),$$

so

$$\text{diam}(A_\omega) \leq 2\eta_0 + \text{diam}(f_{[\omega]_{n_\varepsilon}}(B)) < 3\varepsilon.$$

As ε was arbitrarily chosen, we conclude that $\text{diam}(A_\omega) = 0$.

From (12) we conclude that there exists $a_\omega \in X$ such that $A_\omega = \{a_\omega\}$ and, from (10), we get

$$\lim_{n \rightarrow \infty} h(f_{[\omega]_n}(B), \{a_\omega\}) = 0,$$

for each $B \in \mathcal{K}(X)$.

Note that the above limit is uniform with respect to $\omega \in \Lambda(I)$, i.e.

$$\lim_{n \rightarrow \infty} \sup_{\omega \in \Lambda(I)} h(f_{[\omega]_n}(B), \{a_\omega\}) = 0.$$

Indeed,

$$h(f_{[\omega]_n}(B), \{a_\omega\}) \leq \sum_{k=n}^m h(f_{[\omega]_k}(B), f_{[\omega]_{k+1}}(B)) + h(f_{[\omega]_{m+1}}(B), \{a_\omega\})$$

for every $m, n \in \mathbb{N}$, $m \geq n$. By passing to limit as $m \rightarrow \infty$, we get

$$\begin{aligned} h(f_{[\omega]_n}(B), \{a_\omega\}) &\leq \sum_{k \geq n} h(f_{[\omega]_k}(B), f_{[\omega]_{k+1}}(B)) \stackrel{\text{Remark 2.6}}{\leq} \\ &\leq \sum_{k \geq n} \delta(f_{[\omega]_k}(B), f_{[\omega]_k}(F_S(B))) = \sum_{k \geq n} x_k(B, F_S(B)) \end{aligned}$$

for every $\omega \in \Lambda(I)$ and every $n \in \mathbb{N}$, so

$$\sup_{\omega \in \Lambda(I)} h(f_{[\omega]_n}(B), \{a_\omega\}) \leq \sum_{k \geq n} x_k(B, F_S(B))$$

for every $n \in \mathbb{N}$. As the series $\sum_n x_n(B, F_S(B))$ is convergent, we conclude that $\lim_{n \rightarrow \infty} \sup_{\omega \in \Lambda(I)} h(f_{[\omega]_n}(B), \{a_\omega\}) = 0$.

iii) As

$$\begin{aligned} &h(F_S^{[n]}(B), \{a_\omega \mid \omega \in \Lambda(I)\}) = \\ &= h\left(\bigcup_{\omega \in \Lambda_n(I)} \bigcup_{\alpha \in \Lambda(I)} f_{[\omega\alpha]_n}(B), \bigcup_{\omega \in \Lambda_n(I)} \bigcup_{\alpha \in \Lambda(I)} \{a_{\omega\alpha} \mid \alpha \in \Lambda(I)\}\right) \stackrel{\text{Proposition 2.3}}{\leq} \\ &\leq \sup_{\omega \in \Lambda_n(I)} \sup_{\alpha \in \Lambda(I)} h(f_\omega(B), \{a_{\omega\alpha}\}), \end{aligned}$$

we have

$$\begin{aligned} h(A, \{a_\omega \mid \omega \in \Lambda(I)\}) &\leq h(A, F_S^{[n]}(B)) + h(F_S^{[n]}(B), \{a_\omega \mid \omega \in \Lambda(I)\}) \leq \\ &\leq h(A, F_S^{[n]}(B)) + \sup_{\omega \in \Lambda_n(I)} \sup_{\alpha \in \Lambda(I)} h(f_\omega(B), \{a_{\omega\alpha}\}) \end{aligned} \quad (13)$$

for all $n \in \mathbb{N}^*$ and $B \in \mathcal{K}(X)$.

Since

$$\lim_{n \rightarrow \infty} h(F_S^{[n]}(B), A) = 0$$

(see i)) and

$$\lim_{n \rightarrow \infty} \sup_{\omega \in \Lambda_n(I)} \sup_{\alpha \in \Lambda(I)} h(f_\omega(B), \{a_{\omega\alpha}\}) = 0$$

(see ii)), by passing to limit in (13), we obtain that

$$h(A, \{a_\omega \mid \omega \in \Lambda(I)\}) = 0,$$

i.e.

$$h(A, \overline{\{a_\omega \mid \omega \in \Lambda(I)\}}) = 0$$

(see Proposition 2.2).

Hence

$$A = \overline{\{a_\omega \mid \omega \in \Lambda(I)\}}.$$

iv) Let us consider $(Y_n)_{n \in \mathbb{N}} \subseteq \mathcal{K}(X)$ and $Y \in \mathcal{K}(X)$ such that

$$\lim_{n \rightarrow \infty} h(Y_n, Y) = 0.$$

Using Proposition 2.8 we conclude that

$$H \stackrel{\text{def}}{=} Y \cup \left(\bigcup_{n=0}^{\infty} Y_n \right) \in \mathcal{K}(X).$$

Hence, as the functions f_i are continuous, they are uniformly continuous on H , so for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$d(f_i(x), f_i(y)) < \frac{\varepsilon}{2}$$

for every $i \in I$ and every $x, y \in H$ such that $d(x, y) < \delta_\varepsilon$.

For each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$h(Y_n, Y) < \frac{\delta_\varepsilon}{2}$$

for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

Let us consider $i \in I$ and $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

Since for every $x \in Y_n \subseteq H$ there exists $y \in Y \subseteq H$ such that

$$d(x, y) < d(x, Y) + \frac{\delta_\varepsilon}{2},$$

we get that

$$d(x, y) < h(Y_n, Y) + \frac{\delta_\varepsilon}{2} < \frac{\delta_\varepsilon}{2} + \frac{\delta_\varepsilon}{2} = \delta_\varepsilon,$$

so

$$d(f_i(x), f_i(Y)) \leq d(f_i(x), f_i(y)) < \frac{\varepsilon}{2}.$$

Consequently

$$d(f_i(Y_n), f_i(Y)) \leq \frac{\varepsilon}{2}.$$

In the same manner, one can prove that

$$d(f_i(Y), f_i(Y_n)) \leq \frac{\varepsilon}{2},$$

so

$$h(f_i(Y), f_i(Y_n)) \leq \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned} h(F_S(Y_n), F_S(Y)) &= h\left(\bigcup_{i \in I} f_i(Y_n), \bigcup_{i \in I} f_i(Y)\right) \stackrel{\text{Proposition 2.3}}{\leq} \\ &\leq \max_{i \in I} h(f_i(Y_n), f_i(Y)) \leq \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Thus for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $h(F_S(Y_n), F_S(Y)) < \varepsilon$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, i.e.

$$\lim_{n \rightarrow \infty} h(F_S(Y_n), F_S(Y)) = 0.$$

v) Since $\lim_{n \rightarrow \infty} h(F_S^{[n]}(A), A) = 0$ (see i) for $B = A$), using 4) for $Y_n = F_S^{[n]}(A) \in \mathcal{K}(X)$ and $Y = A \in \mathcal{K}(X)$, we obtain that

$$\lim_{n \rightarrow \infty} h(F_S^{[n+1]}(A), F_S(A)) = 0. \quad (14)$$

Using i), for $B = F_S(A)$, we infer that

$$\lim_{n \rightarrow \infty} h(F_S^{[n+1]}(A), A) = 0. \quad (15)$$

From (14) and (15) we conclude that

$$F_S(A) = A.$$

Moreover, if for some $A_1 \in \mathcal{K}(X)$ we have $F_S(A_1) = A_1$, then $F_S^{[n]}(A_1) = A_1$ for each $n \in \mathbb{N}$, so $\lim_{n \rightarrow \infty} h(F_S^{[n]}(A_1), A_1) = 0$. Since, according to i), we have $\lim_{n \rightarrow \infty} h(F_S^{[n]}(A_1), A) = 0$, we conclude that $A = A_1$. \square

Let us note that, concerning the speed of convergence of the sequence $(F_S^{[n]}(B))_{n \in \mathbb{N}}$, where $B \in \mathcal{K}(X)$, we have (from the proof of i), the following inequality:

$$h(F_S^{[n]}(B), A) \leq \frac{d^{\lfloor \frac{n}{2} \rfloor}}{1-d} (x_0(B, F_S(B)) + x_1(B, F_S(B))),$$

for every $n \in \mathbb{N}$.

Remark 3.3. *By taking in the above Theorem a set I with one element, we get that A has exactly one element which is the fixed point of the convex contraction that can be approximated by means of Picard iteration. Consequently we obtain Istrătescu's fixed point theorem for convex contractions.*

Proposition 3.4. *Let $\mathcal{S} = ((X, d), (f_i)_{i \in I})$ be an iterated function system consisting of convex contractions. Then, in the framework of Theorem 3.2, we have*

$$\lim_{n \rightarrow \infty} \text{diam}(A_{[\omega]_n}) = 0$$

for every $\omega \in \Lambda(I)$.

Proof. Take $B = A$ in (11) from the proof of Theorem 3.2. \square

Proposition 3.5. *Let $\mathcal{S} = ((X, d), (f_i)_{i \in I})$ be an iterated function system consisting of convex contractions. Then, in the framework of Theorem 3.2, we have*

$$\bigcap_{n \in \mathbb{N}} A_{[\omega]_n} = \{a_\omega\}$$

for every $\omega \in \Lambda(I)$.

Proof. From $F_S(A) = A$ we infer that

$$A_{[\omega]_{n+1}} \subseteq A_{[\omega]_n}$$

for every $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} h(A_{[\omega]_n}, \bigcap_{n \in \mathbb{N}} A_{[\omega]_n}) = 0$$

(see Theorem 1.14 from [23]) and taking into account Theorem 3.2, ii), we conclude that $\bigcap_{n \in \mathbb{N}} A_{[\omega]_n} = \{a_\omega\}$. \square

Using the above two Propositions, the same arguments as the ones used in the proof of Theorem 4.1 from [18] give us the following:

Theorem 3.6. *Let $\mathcal{S} = ((X, d), (f_i)_{i \in I})$ be an iterated function system consisting of convex contractions. Then, in the framework of Theorem 3.2, the function $\pi : \Lambda(I) \rightarrow A$ defined by*

$$\pi(\omega) = a_\omega,$$

for every $\omega \in \Lambda$, which is called the canonical projection from $\Lambda(I)$ to A , has the following properties:

- 1) *it is continuous;*
- 2) *it is onto;*
- 3)

$$\pi \circ F_i = f_i \circ \pi,$$

for every $i \in I$.

References

- [1] J. Andres and M. Rypka, Multivalued fractals and hyperfractals, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **22** (2012), DOI 10.1142/S02181127412500095.
- [2] M. Boriceanu, M. Bota and A. Petruşel, Multivalued fractals in b -metric spaces, Cent. Eur. J. Math., **8** (2010), 367-377.
- [3] C. Chifu and A. Petruşel, Multivalued fractals and generalized multivalued contractions, Chaos Solitons Fractals, **36** (2008), 203-210.
- [4] D. Dumitru, Generalized iterated function systems containing Meir-Keeler functions, An. Univ. Bucur., Mat., **58** (2009), 109-121.
- [5] V. Istrăţescu, Some fixed point theorems for convex contraction mappings and convex nonexpansive mappings (I), Libertas Math., **1** (1981), 151-164.
- [6] V. Istrăţescu, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters - I, Annali di Mat. Pura Appl., **130** (1982), 89-104.
- [7] V. Istrăţescu, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters, II, Annali di Mat. Pura Appl., **134** (1983), 327-362.
- [8] V. Ghorbanian, S. Rezapour and N. Shahzad, Some ordered fixed point results and the property (P), Comput. Math. Appl., **63** (2012), 1361-1368.
- [9] G. Gwóźdź-Lukowska and J. Jachymski, IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem, J. Math. Anal. Appl., **356** (2009), 453-463.

- [10] N. Hussain, M. A. Kutbi, S. Khaleghizadeh and P. Salimi, Discussions on recent results for α - Ψ -contractive mappings, *Abstr. Appl. Anal.*, vol. 2014, Article ID 456482, 13 pages, 2014.
- [11] M. Klimek and M. Kosek, Generalized iterated function systems, multifunctions and Cantor sets, *Ann. Polon. Math.*, **96** (2009), 25-41.
- [12] A. Latif, W. Sintunavarat and A. Ninsri, Approximate fixed point theorems for partial generalized convex contraction mappings in α -complete metric spaces, *Taiwanese J. Math.*, **19** (2015), 315-333.
- [13] K. Leśniak, Infinite iterated function systems: a multivalued approach, *Bull. Pol. Acad. Sci. Math.*, **52** (2004), 1-8.
- [14] E. Llorens-Fuster, A. Petruşel and J.-C. Yao, Iterated function systems and well posedness, *Chaos Solitons Fractals*, **41** (2009), 1561-1568.
- [15] L. Máté, The Hutchinson-Barnsley theory for certain noncontraction mappings, *Period. Math. Hungar.*, **27** (1993), 21-33.
- [16] M. A. Miandaragh, M. Postolache and S. Rezapour, Approximate fixed points of generalized convex contractions, *Fixed Point Theory Appl.*, vol. 2013, article 255, 2013.
- [17] A. Mihail and R. Miculescu, Applications of fixed point theorems in the theory of generalized IFS, *Fixed Point Theory Appl.*, 2008, Art. ID 312876, 11 pp.
- [18] A. Mihail and R. Miculescu, The shift space for an infinite iterated function system, *Math. Rep.(Bucur.)*, **61** (2009), 21-31.
- [19] A. Mihail and R. Miculescu, Generalized IFSs on noncompact spaces, *Fixed Point Theory Appl.*, 2010, Art. ID 584215, 15 pp.
- [20] A. Petruşel, Iterated function system of locally contractive operators, *Rev. Anal. Numér. Théor. Approx.*, **33** (2004), 215-219.
- [21] A. Petruşel, Fixed point theory and the mathematics of fractals, *Topics in mathematics, computer science and philosophy*, 165-172, Presa Univ. Clujeană, Cluj-Napoca, 2008.
- [22] N. A. Secelean, Iterated function systems consisting of F -contractions, *Fixed Point Theory Appl.*, 2013, 2013:277.
- [23] N. A. Secelean, Countable iterated function systems, Lambert Academic Publishing, 2013.
- [24] N. A. Secelean, Generalized iterated function systems on the space $l^\infty(X)$, *J. Math. Anal. Appl.*, **410** (2014), 847-458.
- [25] F. Strobin and J. Swaczyna, On a certain generalization of the iterated function system, *Bull. Australian Math. Soc.*, **87** (2013), 37-54.

[26] F. Strobil, Attractors of generalized IFSs that are not attractors of IFSs, J. Math. Anal. Appl., **422** (2015), 99-108.

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